Attitude Estimation

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The discussion here is regarding attitude estimation from raw gyroscope measurements. Three different parameterizations of attitude is considered namely, Euler angles ($\boldsymbol{\theta} \in \mathbb{R}^3$) , Direction Cosine Matrix (DCM) ($\mathtt{R} \in \mathrm{SO}(3)$) and quaternions ($q$). A simple euler integration is considered for attitude estimation and a linear (incremental) covariance propagation is derived. Dynamics governing each of the three parameterizations are;

$$\dot{\mathtt{R}}_{\mathrm{WB}} = \mathtt{R}_{\mathrm{WB}}\ {}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}}^{\wedge}, \qquad \dot{\mathbf{q}} = \frac{1}{2}\mathbf{q} \otimes {}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}} \qquad \dot{\boldsymbol{\theta}} = E(\boldsymbol{\theta})\ {}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}} \tag{1}$$

where $(.)^{\wedge}$ is the hat operator representing skew-symmetric matrix of the input vector and $\otimes$ is the quaternion product. Following gyroscope model is considered,

$${}_{\mathrm{B}}\tilde{\boldsymbol{\omega}}_{\mathrm{WB}}(t) = {}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}}(t) + \mathbf{b}^g(t) + \boldsymbol{\eta}^g(t), \tag{2}$$

where $\mathbf{b}^g(t) \in \mathbb{R}^3$ is bias and $\boldsymbol{\eta}^g(t) \sim \mathcal{N}(\mathbf{0}_3, \boldsymbol{\Sigma}^g)$. Following sections derive attitude as a function of time along with covariance.

Content here is partly from [1]. There is accompanying code written in C++ and can be found at github[1].

1 DCM

1.1 Identities

Before beginning, following identities will be quite handy, Let $\mathbf{f} : \mathbb{R}^3 \to \mathrm{SO}(3)$, then right jacobian $\mathtt{J}_r$ is given by,

$$\mathtt{J}_r(\boldsymbol{\theta}) = \underset{\delta\boldsymbol{\theta} \to \mathbf{0}}{Lt} \frac{\mathbf{f}(\boldsymbol{\theta} + \delta\boldsymbol{\theta}) \ominus \mathbf{f}(\boldsymbol{\theta})}{\delta\boldsymbol{\theta}} \tag{3}$$

From above, $\mathbf{f}(\boldsymbol{\theta} + \delta\boldsymbol{\theta}) = \mathbf{f}(\boldsymbol{\theta}) \oplus \frac{\partial \mathbf{f}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \delta\boldsymbol{\theta}$. When $\mathbf{f}$ is $\mathrm{Exp} : \mathbb{R}^3 \to \mathrm{SO}(3)$, $\mathrm{Exp}(\boldsymbol{\theta} + \delta\boldsymbol{\theta}) = \mathrm{Exp}(\boldsymbol{\theta})\mathrm{Exp}\left(\mathtt{J}_r(\boldsymbol{\theta})\delta\boldsymbol{\theta}\right)$.

The second useful property is,

$$\mathrm{Exp}(\boldsymbol{\theta})\mathtt{R} = \mathtt{R}\mathrm{Exp}(\mathtt{R}^{\mathsf{T}}\boldsymbol{\theta}) \tag{4}$$

1.2 Derivation

Integrating (1), gives,

$$\mathtt{R}_{\mathrm{WB}}(t + \Delta t) = \mathtt{R}_{\mathrm{WB}}(t)\ \mathrm{Exp}\left(\int_t^{t+\Delta t} {}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}}(\tau) d\tau\right) \tag{5}$$

Using Euler integration assuming ${}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}}$ is constant in the interval $[t, t + \Delta t]$,

$$\mathtt{R}_{\mathrm{WB}}(t + \Delta t) = \mathtt{R}_{\mathrm{WB}}(t)\ \mathrm{Exp}\left({}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}}(t)\Delta t\right). \tag{6}$$

Dropping subscript notation and using gyroscope model as stated in (2),

$$\mathtt{R}(t + \Delta t) = \ \mathtt{R}(t)\ \mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}(t) - \mathbf{b}^g(t) - \boldsymbol{\eta}^{gd}(t)\right)\Delta t\right). \tag{7}$$

Writing $\mathtt{R}(t)$ as $\mathtt{R}_i$ and integrating the above equation repeatedly for $k = [i, i+1, \ldots, j-1]$ gives,

$$\mathtt{R}_j = \ \mathtt{R}_i \prod_{k=i}^{j-1} \mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \mathbf{b}_k^g - \boldsymbol{\eta}_k^{gd}\right)\Delta t\right). \tag{8}$$

[1] https://github.com/kvmanohar22/attitude_estimation

The above can be simplified applying (3) in (9) and (4) in (10) repeatedly by moving the noise terms to the far right gives,

$$\mathtt{R}_j = \mathtt{R}_i \prod_{k=i}^{j-1} \mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \mathbf{b}_k^g - \boldsymbol{\eta}_k^{gd}\right)\Delta t\right) \tag{9}$$

$$= \mathtt{R}_i \prod_{k=i}^{j-1} \mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \mathbf{b}_k^g\right)\Delta t\right) \mathrm{Exp}\left(-\mathtt{J}_r\left(\tilde{\boldsymbol{\omega}}_k - \mathbf{b}_k^g\right)\boldsymbol{\eta}_k^{gd}\Delta t\right) \tag{10}$$

$$= \mathtt{R}_i \left(\prod_{k=i}^{j-1} \mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \mathbf{b}_k^g\right)\Delta t\right)\right) \prod_{k=i}^{j-1} \mathrm{Exp}\left(-\prod_{m=k+1}^{j-1} \mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_m - \mathbf{b}_k^g\right)\Delta t\right)^{\mathsf{T}} \mathtt{J}_r\left(\tilde{\boldsymbol{\omega}}_k - \mathbf{b}_k^g\right)\boldsymbol{\eta}_k^{gd}\Delta t\right) \tag{11}$$

The above equation can be simplified by using the shorthand notation, $\Delta\tilde{\mathtt{R}}_{ij} \coloneqq \prod_{k=i}^{j-1} \mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \mathbf{b}_k^g\right)\Delta t\right)$ and $\mathtt{J}_r^k \coloneqq \mathtt{J}_r\left(\tilde{\boldsymbol{\omega}}_k - \mathbf{b}_k^g\right)$,

$$\begin{aligned}
\mathtt{R}_j &= \mathtt{R}_i \left(\prod_{k=i}^{j-1} \mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \mathbf{b}_k^g\right)\Delta t\right)\right) \prod_{k=i}^{j-1} \mathrm{Exp}\left(-\prod_{m=k+1}^{j-1} \mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_m - \mathbf{b}_k^g\right)\Delta t\right)^{\mathsf{T}} \mathtt{J}_r^k \boldsymbol{\eta}_k^{gd}\Delta t\right) \\
&= \mathtt{R}_i \; \Delta\tilde{\mathtt{R}}_{ij} \prod_{k=i}^{j-1} \mathrm{Exp}\left(-\Delta\tilde{\mathtt{R}}_{k+1j}^{\mathsf{T}} \mathtt{J}_r^k \boldsymbol{\eta}_k^{gd}\Delta t\right) \\
&= \mathtt{R}_i \; \Delta\tilde{\mathtt{R}}_{ij} \mathrm{Exp}\left(-\delta\boldsymbol{\phi}_{ij}\right)
\end{aligned} \tag{12}$$

where noise is defined as $\mathrm{Exp}(-\delta\boldsymbol{\phi}_{ij}) \coloneqq \prod_{k=i}^{j-1} \mathrm{Exp}\left(-\Delta\tilde{\mathtt{R}}_{k+1j}^{\mathsf{T}} \mathtt{J}_r^k \boldsymbol{\eta}_k^{gd}\Delta t\right)$ From (12), the noise has been separated and it is easy to read mean of the distribution. Noise can now be further analyzed to obtain an expression for covariance. From the noise definition we have

$$\begin{aligned}
\delta\boldsymbol{\phi}_{ij} &= -\mathrm{Log}\left(\prod_{k=i}^{j-1} \mathrm{Exp}\left(-\Delta\tilde{\mathtt{R}}_{k+1j}^{\mathsf{T}} \mathtt{J}_r^k \boldsymbol{\eta}_k^{gd}\Delta t\right)\right) \\
&\approx -\sum_{k=i}^{j-1} \Delta\tilde{\mathtt{R}}_{k+1j}^{\mathsf{T}} \mathtt{J}_r^k \boldsymbol{\eta}_k^{gd}\Delta t
\end{aligned} \tag{13}$$

where above is obtained by repeated application of $\mathrm{Log}(\mathrm{Exp}(\boldsymbol{\phi})\mathrm{Exp}(\delta\boldsymbol{\phi})) \approx \boldsymbol{\phi} + \mathtt{J}_r^{-1}(\boldsymbol{\phi})\delta\boldsymbol{\phi}$. Up to first order, $\delta\boldsymbol{\phi}_{ij}$ is a linear combination of zero-mean Gaussian noise $\boldsymbol{\eta}_k^{gd}$ and hence $\delta\boldsymbol{\phi}_{ij}$ is also zero-mean Gaussian white noise. (13) gives expression for noise as a function of time but for every new measurement, the entire sum has to be recomputed. But that can be avoided by re-arranging the terms as follows,

$$\begin{aligned}
\delta\boldsymbol{\phi}_{ij} &= -\sum_{k=i}^{j-1} \Delta\tilde{\mathtt{R}}_{k+1j}^{\mathsf{T}} \; \mathtt{J}_r^k \boldsymbol{\eta}_k^{gd}\Delta t \\
&= -\sum_{k=i}^{j-2} \Delta\tilde{\mathtt{R}}_{k+1j}^{\mathsf{T}} \; \mathtt{J}_r^k \boldsymbol{\eta}_k^{gd}\Delta t - \Delta\tilde{\mathtt{R}}_{jj}^{\mathsf{T}} \; \mathtt{J}_r^{j-1} \boldsymbol{\eta}_{j-1}^{gd}\Delta t \\
&= -\sum_{k=i}^{j-2} \Delta\tilde{\mathtt{R}}_{k+1j}^{\mathsf{T}} \; \mathtt{J}_r^k \boldsymbol{\eta}_k^{gd}\Delta t - \mathtt{J}_r^{j-1} \boldsymbol{\eta}_{j-1}^{gd}\Delta t \\
&= -\sum_{k=i}^{j-2} \left(\Delta\tilde{\mathtt{R}}_{k+1j-1} \; \Delta\tilde{\mathtt{R}}_{j-1j}\right)^{\mathsf{T}} \; \mathtt{J}_r^k \boldsymbol{\eta}_k^{gd}\Delta t - \mathtt{J}_r^{j-1} \boldsymbol{\eta}_{j-1}^{gd}\Delta t \\
&= -\sum_{k=i}^{j-2} \Delta\tilde{\mathtt{R}}_{j-1j}^{\mathsf{T}} \; \Delta\tilde{\mathtt{R}}_{k+1j-1}^{\mathsf{T}} \; \mathtt{J}_r^k \boldsymbol{\eta}_k^{gd}\Delta t - \mathtt{J}_r^{j-1} \boldsymbol{\eta}_{j-1}^{gd}\Delta t \\
&= -\Delta\tilde{\mathtt{R}}_{j-1j}^{\mathsf{T}} \left(\sum_{k=i}^{j-2} \; \Delta\tilde{\mathtt{R}}_{k+1j-1}^{\mathsf{T}} \; \mathtt{J}_r^k \boldsymbol{\eta}_k^{gd}\Delta t\right) - \mathtt{J}_r^{j-1} \boldsymbol{\eta}_{j-1}^{gd}\Delta t \\
&= -\Delta\tilde{\mathtt{R}}_{j-1j}^{\mathsf{T}} \delta\boldsymbol{\phi}_{ij-1} - \mathtt{J}_r^{j-1} \boldsymbol{\eta}_{j-1}^{gd}\Delta t
\end{aligned} \tag{14}$$

Noise at t_j is a linear combination of noise at t_{j-1} and the latest measurement. Assuming at the start the noise $\delta\boldsymbol{\phi}_0$ is zero-mean gaussian, $\delta\boldsymbol{\phi}_{ij}$ being linear combination of zero-mean gaussians, is again zero-mean gaussian. Clearly $\delta\hat{\boldsymbol{\phi}}_{ij} = \mathbb{E}[\delta\boldsymbol{\phi}_{ij}] = \mathbf{0}$. Denoting $\delta\boldsymbol{\phi}_{ij} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{ij})$,

$$
\begin{aligned}
\boldsymbol{\Sigma}_{ij} &= \mathbb{E}\left[\left(\delta\boldsymbol{\phi}_{ij} - \delta\hat{\boldsymbol{\phi}}_{ij}\right)\left(\delta\boldsymbol{\phi}_{ij} - \delta\hat{\boldsymbol{\phi}}_{ij}\right)^{\mathsf{T}}\right] \\
&= \mathbb{E}\left[\delta\boldsymbol{\phi}_{ij}\ \delta\boldsymbol{\phi}_{ij}^{\mathsf{T}}\right] \\
&= \mathbb{E}\left[\left(-\Delta\tilde{\mathtt{R}}_{j-1j}^{\mathsf{T}}\delta\boldsymbol{\phi}_{ij-1} - \mathtt{J}_r^{j-1}\boldsymbol{\eta}_{j-1}^{gd}\Delta t\right)\ \left(-\Delta\tilde{\mathtt{R}}_{j-1j}^{\mathsf{T}}\delta\boldsymbol{\phi}_{ij-1} - \mathtt{J}_r^{j-1}\boldsymbol{\eta}_{j-1}^{gd}\Delta t\right)^{\mathsf{T}}\right] \\
&= \Delta\tilde{\mathtt{R}}_{j-1j}^{\mathsf{T}}\mathbb{E}\left[\delta\boldsymbol{\phi}_{ij-1}\delta\boldsymbol{\phi}_{ij-1}^{\mathsf{T}}\right]\Delta\tilde{\mathtt{R}}_{j-1j}^{\mathsf{T}} + \mathtt{J}_r^{j-1}\mathbb{E}\left[\boldsymbol{\eta}_{j-1}^{gd}\boldsymbol{\eta}_{j-1}^{gd\ \mathsf{T}}\right]\mathtt{J}_r^{j-1\,\mathsf{T}}\Delta t^2 \\
&= \Delta\tilde{\mathtt{R}}_{j-1j}^{\mathsf{T}}\boldsymbol{\Sigma}_{ij-1}\Delta\tilde{\mathtt{R}}_{j-1j}^{\mathsf{T}} + \mathtt{J}_r^{j-1}\boldsymbol{\Sigma}^{gd}\mathtt{J}_r^{j-1\,\mathsf{T}}\Delta t^2,
\end{aligned}
\tag{15}
$$

where variables $\boldsymbol{\eta}_{j-1}^{gd}$ and $\delta\boldsymbol{\phi}_{ij-1}$ are assumed to be uncorrelated.
Note how inefficient (13) is as illustrated in Fig. 1.

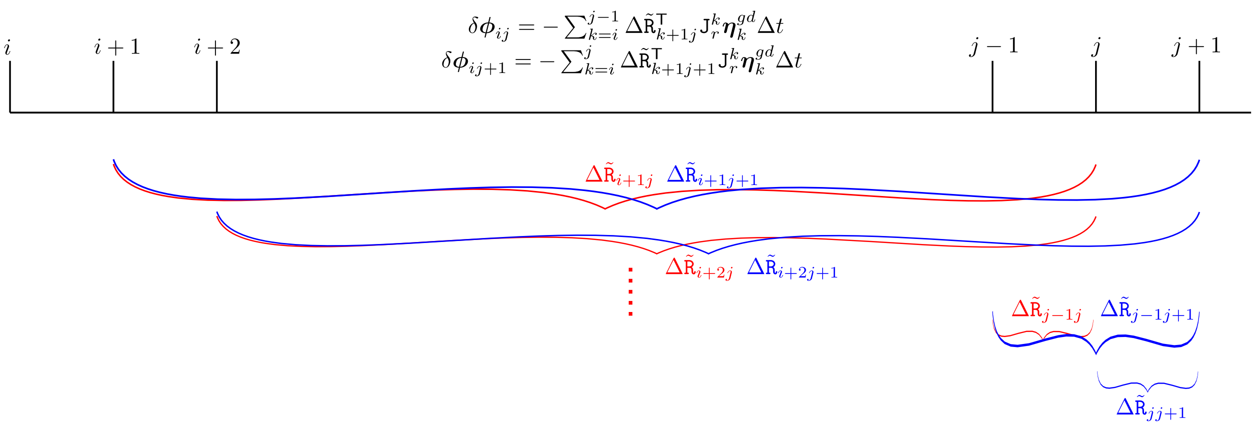

Fig. 1: Linear Propagation Covariance computation inefficiency when using (13). The ones in red are computed upto time t_j and with a new measurement at t_{j+1}, every delta measurement needs to be recomputed and worst they need to be stored.

2 Euler angles

2.1 Derivation of dynamics equation

Let $\boldsymbol{\theta} = (\alpha \ \ \beta \ \ \gamma)^{\mathsf{T}}$ be euler angles that represent attitude of body frame (B) in world frame (W). Using zyx rotation convention,

$$
\begin{aligned}
\mathtt{R}_{\mathrm{WB}} &= \mathtt{R}_3(-\gamma)\mathtt{R}_2(-\beta)\mathtt{R}_1(-\alpha) \\
&= \mathtt{R}_{\mathrm{W}b_2}\ \mathtt{R}_{b_2b_1}\ \mathtt{R}_{b_1\mathrm{B}}
\end{aligned}
\tag{16}
$$

where additional axes b_1 and b_2 are introduced. Using the identity of angular velocities,

$$
\begin{aligned}
{}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}} &= {}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{W}b_2} + {}_{\mathrm{B}}\boldsymbol{\omega}_{b_2b_1} + {}_{\mathrm{B}}\boldsymbol{\omega}_{b_1\mathrm{B}} \\
&= \mathtt{R}_{b_1\mathrm{B}}^{\mathsf{T}}\ \mathtt{R}_{b_2b_1}^{\mathsf{T}}\ {}_{b_2}\boldsymbol{\omega}_{\mathrm{W}b_2} + \mathtt{R}_{b_1\mathrm{B}}^{\mathsf{T}}\ {}_{b_1}\boldsymbol{\omega}_{b_2b_1} + {}_{\mathrm{B}}\boldsymbol{\omega}_{b_1\mathrm{B}} \\
&= \mathtt{R}_1(\alpha)\ \mathtt{R}_2(\beta)\ {}_{b_2}\boldsymbol{\omega}_{\mathrm{W}b_2} + \mathtt{R}_1(\alpha)\ {}_{b_1}\boldsymbol{\omega}_{b_2b_1} + {}_{\mathrm{B}}\boldsymbol{\omega}_{b_1\mathrm{B}}
\end{aligned}
\tag{17}
$$

Further, angular velocities can be written as,

$$
{}_{\mathrm{B}}\boldsymbol{\omega}_{b_1\mathrm{B}} = (\dot{\alpha} \ \ 0 \ \ 0)^{\mathsf{T}} \qquad {}_{b_1}\boldsymbol{\omega}_{b_2b_1} = (0 \ \ \dot{\beta} \ \ 0)^{\mathsf{T}} \qquad {}_{b_2}\boldsymbol{\omega}_{\mathrm{W}b_2} = (0 \ \ 0 \ \ \dot{\gamma})^{\mathsf{T}}
\tag{18}
$$

Substituting (16), (18) in (17), and after simplication,

$$
{}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}} = \begin{pmatrix} 1 & 0 & -\sin(\beta) \\ 0 & \cos(\alpha) & \sin(\alpha)\cos(\beta) \\ 0 & -\sin(\alpha) & \cos(\alpha)\cos(\beta) \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}
\tag{19}
$$

Inverting the above gives the familiar euler rate equation,

$$
\dot{\boldsymbol{\theta}} := \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} 1 & \sin\alpha\tan\beta & \cos\alpha\tan\beta \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha\sec\beta \end{pmatrix} {}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}}
\tag{20}
$$

2.2 Euler Angle State Estimation

In this case euler angles ($\boldsymbol{\theta} = (\alpha \quad \beta \quad \gamma)^{\mathsf{T}}$) are integrated according to (1) where

$$E(\boldsymbol{\theta}) := \begin{pmatrix} 1 & \sin\alpha\tan\beta & \cos\alpha\tan\beta \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha\sec\beta \end{pmatrix} \tag{21}$$

Note the singularity as $\beta \to 90°$.

$$\boldsymbol{\theta}(t+\Delta t) = \boldsymbol{\theta}(t) + \int_t^{t+\Delta t} E(\boldsymbol{\theta}(\tau))\ {}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}}(\tau)d\tau \tag{22}$$

$$\tag{23}$$

Using euler integration from $[t,\ t+\Delta t]$ gives,

$$\begin{aligned} \boldsymbol{\theta}_{i+1} &= \boldsymbol{\theta}_i + E(\boldsymbol{\theta}_i)\ \boldsymbol{\omega}_i \Delta t \\ &= \boldsymbol{\theta}_i + E(\boldsymbol{\theta}_i)\left(\tilde{\boldsymbol{\omega}}_i - \mathbf{b}_i^g - \boldsymbol{\eta}_i^{gd}\right)\Delta t \end{aligned} \tag{24}$$

RHS of (24) is non-linear, we linearize about the current mean estimate (i.e, $\hat{\boldsymbol{\theta}} = \mathbb{E}[\boldsymbol{\theta}]$) and retain terms up to first order,

$$\begin{aligned} \boldsymbol{\theta}_{i+1} &= \mathbf{f}(\boldsymbol{\theta}_i, \boldsymbol{\eta}_i^{gd}) \\ &\approx \mathbf{f}(\hat{\boldsymbol{\theta}}_i, \hat{\boldsymbol{\eta}}_i^{gd}) + \frac{\partial \mathbf{f}(\boldsymbol{\theta}_i, \boldsymbol{\eta}_i^{gd})}{\partial \boldsymbol{\theta}_i}\left(\boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}_i\right) + \frac{\partial \mathbf{f}(\boldsymbol{\theta}_i, \boldsymbol{\eta}_i^{gd})}{\partial \boldsymbol{\eta}_i^{gd}}\left(\boldsymbol{\eta}_i^{gd} - \hat{\boldsymbol{\eta}}_i^{gd}\right) \\ &= \mathbf{f}(\hat{\boldsymbol{\theta}}_i, \mathbf{0}) + \frac{\partial \mathbf{f}(\boldsymbol{\theta}_i, \boldsymbol{\eta}_i^{gd})}{\partial \boldsymbol{\theta}_i}\left(\boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}_i\right) + \frac{\partial \mathbf{f}(\boldsymbol{\theta}_i, \boldsymbol{\eta}_i^{gd})}{\partial \boldsymbol{\eta}_i^{gd}}\boldsymbol{\eta}_i^{gd} \end{aligned} \tag{25}$$

where we are using the fact that $\hat{\boldsymbol{\eta}}_i^{gd} = \mathbf{0}$. Applying the above gives,

$$\boldsymbol{\theta}_{i+1} = \hat{\boldsymbol{\theta}}_i + E(\hat{\boldsymbol{\theta}}_i)\left(\tilde{\boldsymbol{\omega}}_i - \mathbf{b}_i^g\right)\Delta t + \left(\mathbf{I}_{3\times3} + \frac{\partial E(\boldsymbol{\theta}_i)\left(\tilde{\boldsymbol{\omega}}_i - \mathbf{b}_i^g\right)\Delta t}{\partial \boldsymbol{\theta}}\right)\left(\boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}_i\right) - E(\hat{\boldsymbol{\theta}}_i)\boldsymbol{\eta}_i^{gd}\Delta t \tag{26}$$

Taking expectation of (26) gives the mean of estimate.

$$\begin{aligned} \hat{\boldsymbol{\theta}}_{i+1} &= \mathbb{E}[\boldsymbol{\theta}_{i+1}] \\ &= \hat{\boldsymbol{\theta}}_i + E(\hat{\boldsymbol{\theta}}_i)\left(\tilde{\boldsymbol{\omega}}_i - \mathbf{b}_i^g\right)\Delta t \end{aligned} \tag{27}$$

Covariance is derived as follows,

$$\begin{aligned} \boldsymbol{\Sigma}_{i+1} &= \mathbb{E}\left[\left(\boldsymbol{\theta}_{i+1} - \hat{\boldsymbol{\theta}}_{i+1}\right)\left(\boldsymbol{\theta}_{i+1} - \hat{\boldsymbol{\theta}}_{i+1}\right)^{\mathsf{T}}\right] \\ &= \mathbb{E}\left[\left(\hat{\boldsymbol{\theta}}_i + E(\hat{\boldsymbol{\theta}}_i)\left(\tilde{\boldsymbol{\omega}}_i - \mathbf{b}_i^g\right)\Delta t + \left(\mathbf{I}_{3\times3} + \frac{\partial E(\boldsymbol{\theta}_i)\left(\tilde{\boldsymbol{\omega}}_i - \mathbf{b}_i^g\right)\Delta t}{\partial \boldsymbol{\theta}}\right)\left(\boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}_i\right) - E(\hat{\boldsymbol{\theta}}_i)\boldsymbol{\eta}_i^{gd}\Delta t - \hat{\boldsymbol{\theta}}_{i+1}\right)\right] \\ &= \mathbb{E}\left[\left(\left(\mathbf{I}_{3\times3} + \frac{\partial E(\boldsymbol{\theta}_i)\left(\tilde{\boldsymbol{\omega}}_i - \mathbf{b}_i^g\right)\Delta t}{\partial \boldsymbol{\theta}}\right)\left(\boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}_i\right) - E(\hat{\boldsymbol{\theta}}_i)\boldsymbol{\eta}_i^{gd}\Delta t\right)(\dots)^{\mathsf{T}}\right] \\ &= \left(\mathbf{I}_{3\times3} + \frac{\partial E(\boldsymbol{\theta}_i)\left(\tilde{\boldsymbol{\omega}}_i - \mathbf{b}_i^g\right)\Delta t}{\partial \boldsymbol{\theta}}\right)\mathbb{E}\left[\left(\boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}_i\right)\left(\boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}_i\right)^{\mathsf{T}}\right]\left(\mathbf{I}_{3\times3} + \frac{\partial E(\boldsymbol{\theta}_i)\left(\tilde{\boldsymbol{\omega}}_i - \mathbf{b}_i^g\right)\Delta t}{\partial \boldsymbol{\theta}}\right)^{\mathsf{T}} \\ &\quad + E(\hat{\boldsymbol{\theta}}_i)\mathbb{E}\left[\boldsymbol{\eta}_i^{gd}\boldsymbol{\eta}_i^{gd\mathsf{T}}\right]E(\hat{\boldsymbol{\theta}}_i)^{\mathsf{T}}\Delta t^2 \\ &= \left(\mathbf{I}_{3\times3} + \frac{\partial E(\boldsymbol{\theta}_i)\left(\tilde{\boldsymbol{\omega}}_i - \mathbf{b}_i^g\right)\Delta t}{\partial \boldsymbol{\theta}}\right)\boldsymbol{\Sigma}_i\left(\mathbf{I}_{3\times3} + \frac{\partial E(\boldsymbol{\theta}_i)\left(\tilde{\boldsymbol{\omega}}_i - \mathbf{b}_i^g\right)\Delta t}{\partial \boldsymbol{\theta}}\right)^{\mathsf{T}} + E(\hat{\boldsymbol{\theta}}_i)\boldsymbol{\Sigma}^{gd}E(\hat{\boldsymbol{\theta}}_i)^{\mathsf{T}}\Delta t^2 \end{aligned} \tag{28}$$

3 Quaternions

3.1 Integration

Taylor expansion of $\mathbf{q}(t+\Delta t)$ is,

$$\mathbf{q}(t+\Delta t) = \mathbf{q}(t) + \dot{\mathbf{q}}(t)\Delta t + \frac{1}{2!}\ddot{\mathbf{q}}(t)\Delta t^2 + \frac{1}{3!}\ddot{\mathbf{q}}(t)\Delta t^3 + \ \dots \tag{29}$$

Repeatedly differentiating (1) to obtain higher order derivatives of $q(t)$ and substituting them back above in (29) gives,

$$\begin{aligned}
\mathbf{q}(t+\Delta t) &= q(t) + \left(\frac{1}{2}q(t)\otimes\boldsymbol{\omega}\right)\Delta t + \frac{1}{2!}\left(\frac{1}{4}q(t)\otimes\boldsymbol{\omega}\otimes\boldsymbol{\omega}\right)\Delta t^2 + \frac{1}{3!}\left(\frac{1}{8}q(t)\otimes\boldsymbol{\omega}\otimes\boldsymbol{\omega}\otimes\boldsymbol{\omega}\right)\Delta t^3 + \ \dots \\
&= q(t)\otimes\left(1 + \left(\frac{1}{2}\boldsymbol{\omega}\Delta t\right) + \frac{1}{2!}\left(\frac{1}{4}\boldsymbol{\omega}\otimes\boldsymbol{\omega}\right)\Delta t^2 + \frac{1}{3!}\left(\frac{1}{8}\boldsymbol{\omega}\otimes\boldsymbol{\omega}\otimes\boldsymbol{\omega}\right)\Delta t^3 + \ \dots\right) \\
&= q(t)\otimes\left(\left\{1 - \frac{1}{2!}\left(\frac{1}{4}||\boldsymbol{\omega}||^2\Delta t^2\right) + \frac{1}{4!}\left(\frac{1}{16}||\boldsymbol{\omega}||^4\Delta t^4\right) - \ \dots\right\} + \left\{\left(\frac{1}{2}\boldsymbol{\omega}\Delta t\right) - \frac{1}{3!}\left(\frac{1}{8}||\boldsymbol{\omega}||^3\boldsymbol{\omega}\Delta t^3\right) + \ \dots\right\}\right) \\
&= q(t)\otimes\left(\left\{1 - \frac{1}{2!}\left(\frac{||\boldsymbol{\omega}||\Delta t}{2}\right)^2 + \frac{1}{4!}\left(\frac{||\boldsymbol{\omega}||\Delta t}{2}\right)^4 - \ \dots\right\} + \frac{\boldsymbol{\omega}}{||\boldsymbol{\omega}||}\left\{\left(\frac{||\boldsymbol{\omega}||\Delta t}{2}\right) - \frac{1}{3!}\left(\frac{||\boldsymbol{\omega}||\Delta t}{2}\right)^3 + \ \dots\right\}\right) \\
&= q(t)\otimes\left(\cos\left(\frac{||\boldsymbol{\omega}||\Delta t}{2}\right) + \frac{\boldsymbol{\omega}}{||\boldsymbol{\omega}||}\sin\left(\frac{||\boldsymbol{\omega}||\Delta t}{2}\right)\right)
\end{aligned}$$

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3.2 Covariance Propagation

TODO

References

1. Forster, C., Carlone, L., Dellaert, F., Scaramuzza, D.: On-manifold preintegration for real-time visual–inertial odometry. Trans. Rob. **33**(1) (February 2017) 1–21